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INDEPENDENT SEQUENCES IN BANACH SPACES

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ABSTRACT
In every ∞-dimensional separable Banach space X there is a fundamental sequence such that no subsequence of it, which is fundamental in X, is independent ("$\{x_n\}$ is fundamental in X" means $X = \overline{\operatorname{span}}\{x_n\}$).

A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called ω-independent if for any sequence of scalars $\{c_n\}_{n=1}^{\infty}$,

$$\sum_{n=1}^{\infty} c_n x_n = 0 \quad \text{implies} \quad c_n = 0 \text{ for every } n.$$

Erdös and Straus proved in [1] that every algebraically independent sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space contains an infinite subsequence $\{x_{n_i}\}$ which is ω-independent. A natural question arises whether these n_i can be chosen so that

$$\overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty} = \overline{\operatorname{span}}\{x_{n_i}\}_{i=1}^{\infty}.$$

In this note we provide a negative answer to this question. We prove the following

THEOREM. *Let X be an infinite dimensional separable Banach space. There exists an algebraically independent sequence $\{f_n\}_{n=1}^{\infty}$ in X which is fundamental in X, such that no fundamental subsequence of $\{f_n\}$ is ω-independent.*

PROOF. Let $\{(e_n^*, e_n)\}_{n=1}^{\infty}$ with $e_n^* \in X^*$, $e_n \in X$, be a total and fundamental biorthogonal system in X, with $\|e_n\| = 1$ for $n = 1, 2, \cdots$.

For $f \in X$ we denote

$$f(n) = e_n^*(f), \qquad \operatorname{supp} f = \{n : f(n) \neq 0\}.$$

We define a sequence $y_1, y_2, \cdots \subset X$ by recursion:

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$$y_1 = 0 \qquad \text{and for } n = 1, 2, \cdots,$$
$$y_{2n} = y_n + 3^{-n} e_n, \qquad y_{2n+1} = -y_n + 3^{-n} e_n$$

and we set

$$f_n = y_n + 3^{-n} e_n.$$

Let us observe that $\operatorname{span}\{f_n\}_{n=1}^m = \operatorname{span}\{e_n\}_{n=1}^m$ for every m, therefore $\{f_n\}_{n=1}^\infty$ is fundamental in X.

We also see that $\|f_m\| < 1$ for every m.

Suppose that $A \subset N$ is such that

$$\overline{\operatorname{span}}\{f_n : n \in A\} = X.$$

We are going to find some coefficients b_n, for $n \in A$, not all of them equal to 0, so that $\Sigma_{n \in A} b_n f_n = 0$, thus proving our theorem.

Let us start with the following observation.

For every m the set

$$A_m \stackrel{\text{def}}{=} \{n \in A : f_n(m) \neq 0\}$$

is infinite.

Indeed A_m is not empty (because e_m^* vanishes on $\operatorname{span}\{f_j : j \notin A_m\}$), hence A_m is infinite (since $A_m \supseteq A_{2m} \cup A_{2m+1}$ and so on), for every m.

Now we shall pick some numbers $\alpha(j, m)$, $\beta(j, m)$ for $m = 1, 2, \cdots$ and $j = 1, 2, \cdots, 3^{m-1}$ so that

$$\begin{aligned} &\beta(3^{m-2}, m-1) < \alpha(1, m) < \beta(1, m) < \alpha(2, m) < \cdots < \\ &\qquad < \alpha(3^{m-1}, m) < \beta(3^{m-1}, m) < \alpha(1, m+1), \end{aligned} \tag{1}$$

$$\alpha(j, m) \in A_{2m}, \quad \beta(j, m) \in A_{2m+1} \qquad \text{for all } j. \tag{2}$$

It is possible to find $\alpha(j, m)$, $\beta(j, m)$ as above, because the sets A_{2m} and A_{2m+1} are infinite.

Let us notice that

$$\begin{aligned} f_{\alpha(j,m)} &= f_{\alpha(j,m)}(m) \cdot (3^m y_m + e_m + g_{\alpha(j,m)}), \\ f_{\beta(j,m)} &= f_{\beta(j,m)}(m) \cdot (-3^m y_m + e_m + g_{\beta(j,m)}), \end{aligned} \tag{3}$$

where $\operatorname{supp} g_{\alpha(j,m)} \cup \operatorname{supp} g_{\beta(j,m)} \subset [2m, \infty)$.

Now we are ready to define the coefficients b_n. If n is not of the form $\alpha(j, m)$ or $\beta(j, m)$, then we set $b_n = 0$. Otherwise, we define b_n by recursion in the following way:

$$b_{\alpha(1,1)} = \frac{1}{2}\,(f_{\alpha(1,1)}(1))^{-1}, \qquad b_{\beta(1,1)} = -\frac{1}{2}\,(f_{\beta(1,1)}(1))^{-1}$$

and, having defined b_n for $n \leqq \beta(3^{m-2}, m-1)$, we set

$$x_m = \sum_{n \leqq \beta(3^{m-2}, m-1)} b_n f_n$$

and, for n of the form $\alpha(j, m)$ or $\beta(j, m)$, we set

$$b_n = \frac{1}{2}\,3^{-m+1} x_m(m) \cdot f_n(m)^{-1}.$$

Let us notice that, for an n like above,

$$|\,b_n\,| = \frac{3}{2} \cdot |\,x_m(m)\,|. \tag{4}$$

For $1 \leqq N \leqq 3^{m-1}$ let us denote

$$B_{N,m} = \sum_{j=1}^{N} (b_{\alpha(j,m)} f_{\alpha(j,m)} + b_{\beta(j,m)} f_{\beta(j,m)}).$$

LEMMA. (a) $B_{N,m}(i) = 0$ *for every* $i < m$, *every* N;
(b) $x_m(i) = 0$ *for every* $i < m$;
(c) $|\,x_m(i)| \leqq 2^{m-1} \cdot 3^{1-i}$ *for every* $i \geqq m$, *every* m;
(d) $|\,B_{N,m}(i)| \leqq 2^{m-1} 3^{1-i}$ *for every* $i \geqq m$, *every* m.

PROOF. By (3), it follows that

$$b_{\alpha(j,m)} f_{\alpha(j,m)} + b_{\beta(j,m)} f_{\beta(j,m)} = -3^{-m+1} x_m(m)(e_m + h_{m,j}),$$

where $\operatorname{supp} h_{m,j} \subset [2m, \infty)$. This yields immediately (a). Moreover,

$$\sum_{j=1}^{3m-1} (b_{\alpha(j,m)} f_{\alpha(j,m)} + b_{\beta(j,m)} f_{\beta(j,m)}) = -x_m(m) e_m + G_m,$$

where $\operatorname{supp} G_m \subset [2m, \infty)$. Hence $x_{m+1}(i) = x_m(i)$ if $i < m$ and $x_{m+1}(m) = 0$, therefore (b) follows by induction, since we have also $x_2(1) = 0$.

By (4), we have for every $k \geqq 2$

$$\begin{aligned} B_{N,k}(i) &\leqq \sum_{j=1}^{N} (|\,b_{\alpha(j,k)} f_{\alpha(j,k)}(i)| + |\,b_{\beta(j,k)} f_{\beta(j,k)}(i)|) \\ &\leqq 3^{-i} \sum_{j=1}^{N} (|\,b_{\alpha(j,k)}| + |\,b_{\beta(j,k)}|) = 3^{-i} \cdot 2N \cdot \frac{3}{2} \cdot |\,x_k(k)| \\ &\leqq 3^{k-i}\,|\,x_k(k)|. \end{aligned}$$

Let us set $x_1(1)=1$. Then the above inequality is valid for $k=1$ as well. Hence

$$|x_m(i)| \leqq \sum_{k=1}^{m-1} |B_{3^{k-1},k}(i)| \leqq 3^{-i} \sum_{k=1}^{m-1} 3^k |x_k(k)|$$

and, in particular,

$$|x_m(m)| \leqq 3^{-m} \sum_{k=1}^{m-1} 3^k |x_k(k)|.$$

By induction on m we obtain

$$|x_m(m)| \leqq \left(\frac{2}{3}\right)^{m-1} \quad \text{and} \quad \sum_{k=1}^{m-1} 3^k |x_k(k)| \leqq 3 \cdot 2^{m-1}$$

and (c), (d) follow easily from the previous inequalities.

Now we are ready to complete the proof of the theorem. Obviously, $b_{\alpha(1,1)} \neq 0$. We shall see that, nevertheless, $\sum_{m \in A} b_m f_m = 0$.

Let us consider a partial sum $S_M = \sum_{n \in A, n \leqq M} b_n f_n$.

Obviously, there exist (unique) $m = m(M)$ and $N = N(M)$ where $N \leqq 3^m$ such that

$$S_M = x_m + B_{N,m} \quad \text{or} \quad S_M = x_m + B_{N,m} + b_m f_M$$

(the first case corresponds to $M = \alpha(N, m)$, the second to $M = \beta(N, m)$).

When $M \to \infty$, then, clearly, $m \to \infty$. From our lemma we derive easily that

$$\|x_m\| < \sum_{i=m}^{\infty} |x_m(i)| < 2 \cdot \left(\frac{2}{3}\right)^{m-1}, \qquad \|B_{N,m}\| < 2 \cdot \left(\frac{2}{3}\right)^{m-1}$$

and by (4),

$$\|b_M f_M\| < \left(\frac{2}{3}\right)^{m-2}.$$

This proves that $S_M \to 0$.

REFERENCE

1. P. Erdös and E. G. Straus, *On linear independence of sequences in a Banach space*, Pacific J. Math. 3 (1953), 689–694.

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